A solution of nonlinear equation for the gravity wave spectra in middle atmosphere from decomposition method

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1. Introduction

Atmospheric gravity waves are important in the study of atmospheric circulation, structure and variability. Although there are effects in the lower atmosphere, the major wave influences occur in the middle atmosphere, between 10 and 110 km altitudes because of the decreasing in air density and the increasing wave amplitudes with altitude (Fritts, et al., 2003). Atmospheric gravity waves contribute to the energy and momentum transport and turbulence production (Nappo, 2002; Zilitinkevich et al., 2009). A more recent work (Tjernstrom et al., 2009, Meillier et al., 2008) suggested that gravity waves are one source of turbulence in Stable Planetary Boundary Layer and Residual Planetary Boundary Layer. In literature, the general structure for the study of gravity waves is the linear wave theory and the Taylor-Goldstein equation is their main governing expression (Gossard and Hooke, 1975). The Taylor-Goldstein equation is obtained from the linearization of the primitive set of equations for an inviscid, non-rotating fluid. Chimonas (2002) and Meillier et al. (2008) analyze some properties of gravity waves in the Stable Boundary Layer from the Taylor-Goldstein equation. Another method of studying the properties of gravity waves is through power spectra. Models of gravity wave spectrum have evolved with time. Various theories constrain gravity wave spectrum to behave in a particular manner over some range of wave numbers or frequencies. These observational and theoretical constraints have led to a canonical gravity wave spectrum that offers insights into mean properties of the gravity wave field and its variations with altitude (Balsley and Carter, 1982; Tsuda et al., 1989). It is also important to mention that most pollutants are emitted or chemically produced within the Stable Planetary Boundary Layer and Residual Planetary Boundary Layer and its evolution plays an important role in determining pollutant dispersion pathways and the chemical properties of atmospheric pollutants (Salmond and McKendry, 2005).

Gravity waves parameterizations are critical components of virtually all large-scale atmospheric models. Aside from the theoretical deficiencies, even the most powerful available computing architectures still cannot run typical NWP (Numerical Weather Prediction) or climate models fast enough to resolve all relevant scales of atmospheric motion. At present, global models must, in practice, be run with horizontal resolutions that cannot typically resolve atmospheric phenomena shorter than ~10-100 km or greater for weather prediction and ~100-1000 km or greater for climate prediction. Many atmospheric processes have shorter horizontal scales than these and some of these "subgrid-scale" processes interact with and affect the larger-scale atmosphere in important ways. Since they cannot be resolved, large-scale models must resort to "parameterizations" that capture the salient effects on the resolved atmosphere. Atmospheric gravity waves are one such unresolved.

From the definition of wave momentum flux deposition produced by a harmonic, Medvedev and Klaassen (1995, 2000) obtained an equation for the gravity wave and present a spectral parameterization scheme for calculating gravity wave momentum deposition in the middle atmosphere. The equation obtained is a differential nonlinear equation and it is not solved directly. The height of layer is discretized in sub-intervals in such a manner that inside each interval a linear equation is resolved considering average value of the quantities.

In this work, we propose an analytical solution for the gravity wave equation. This equation is solved directly without linearization by the decomposition method (Adomian, 1990, 1994a). So, the nonlinear nature of the problem is preserved. Therefore, the errors found are only due to the parameterization used. The results are compared with the continuous linear solution.

2. A solution of nonlinear equation for the gravity wave spectra

An equation for the evolution of gravity wave spectra with height z was deduced by Medvedev and Klaassen (1995, 2000) from of the momentum flux divergence,

$$\frac{\partial S(m_R,z)}{\partial z}-\left(-\frac{\rho_{0z}}{\rho_0}+\frac{m_{Rz}}{m_R}-\beta(m_R,z)\right)S(m_R,z)=0 \tag{1}$$

where S is the power-spectral density of horizontal wind associated with gravity waves at height z, ρ_0 is the mean density of air in reference height, ρ_{0z} is the density of air in height z, m_R is the real part of vertical wavenumber ($m=m_R+im_I$), m_{Rz} is the wavenumber associated with the maximum of gravity wave spectra and β is the coefficient of nonlinear damping due to interactions of the component m_R with other waves in the spectrum.

To solve the Eq. (1) is necessary parameterize the coefficient β, because it is function of power-spectral density S. A parameterization for the coefficient β was suggested by Medvedev and Klaassen (1995, 2000),

$$\beta(m_R,z)=\frac{\sqrt{2\pi}N(z)}{\sigma_u(z)}\exp\left(-\frac{N^2(z)}{2m_R\sigma_u^2(z)}\right) \tag{2}$$

where N is the Brunt-Väisälä frequency and

$$\sigma_u^2\left(z\right)=\int_{m_R}^{\infty}S(m_R',z)dm_R' \tag{3}$$

is the horizontal wind variance created by all waves in the spectrum with vertical wavenumbers larger than the given m_R.

Substituting (2) and (3) in Eq. (1) is obtained,

$$\frac{\partial S(m_R,z)}{\partial z}+\left(\frac{\rho_{0z}}{\rho_0}-\frac{m_{Rz}}{m_R}\right)S(m_R,z)+\frac{\sqrt{2\pi}N(z)}{\sqrt{\int_{m_R}^{\infty}S(m_R,z)dm_R}}\exp\left(-\frac{N^2(z)}{2m_R\int_{m_R}^{\infty}S(m_R,z)dm_R}\right)S(m_R,z)=0 \tag{4}$$

The evolution of gravity wave spectra with height is given by Eq. (4). It is a nonlinear integro-differential equation that has not a simple analytical solution. To solve the Eq. (4) is considered the Adomian's decomposition method (Adomian, 1990, 1994a),

$$S\left(m_R,z\right)=\sum_{n=0}^{8}u_n \tag{5}$$

where,

$$u_n=-\left[\left(\frac{\rho_{0z}}{\rho_0}-\frac{m_{Rz}}{m_R}\right)\mathbf{L}^{-1}u_{n-1}+\mathbf{L}^{-1}\mathrm{A}_{\text{n-1}}\right] \tag{6}$$

with $\mathbf{L}^{-1} \to \int dz$ and $\mathrm{A_n}$ are the Adomian's polynomials (Adomian, 1990, 1994a).

4. Results and discussion

As discussed in the introduction, in the work of Medvedev and Klaassen (1995, 2000) the equation for the gravity wave spectra was solved in discrete layers by a linear approximation. Now, we compare the solution obtained in this work (Eqs. 5 and 6), which considers the nonlinearity of the equation, with the linear solution obtained when it considers the average values of horizontal wind variance $\int_{m_R}^{\infty} S(m_R, z) dm_R$ appearing in Eq. (4). The terms u_n are calculated from Eq. (6).

The solution of Eq. (4) requires the value of gravity wave spectra in a reference height. We assume that the gravity wave spectra consists of a general Desaubies spectrum form (Desaubies, 1976) in $z = 0\ \left(S\left(m_R, 0\right) = S_0\right)$,

$$S_0 = a_0 \frac{N^2}{m_*^3} \frac{m_R/m_*}{1+\left(m_R/m_*\right)^4} \tag{7}$$

where a_0 is a constant, m_* is the wavenumber associated with the maximum of gravity wave spectra, N is Brunt-Väisälä frequency. For comparison of gravity wave spectra calculated from expressions (5 and 6) with the linear solution of Eq. (4) we consider the values used by Medvedev and Klaassen (2000) for a_0, m_* and N, namely $a_0 = 1/6$, $m_* = 0.006 m^{-1}$ and $N = 0.02 s^{-1}$. In Eq. (4) is considered $m_{Rz}/m_R = m_R/\sqrt{R_i}$, where R_i is Richardson flux number. In this case $R_i = 1$. Considering the initial spectra given by Eq. (7) it is possible to calculate analytically the integrals of Eq. (7),

$$\int_{m_R}^{\infty} a_0 \frac{N^2}{m_*^3} \frac{m_R/m_*}{1+\left(m_R/m_*\right)^4} dm_R = a_0 \frac{N^2}{2m_*^2} \arctan\left(\frac{m_*^2}{m_R^2}\right) \quad \text{for } m_R > 0 \tag{8}$$

Substituting Eq. (8) in Eq. (6), we obtain a simple algebraic expression that, although long, requires a time machine extremely small. Figure 1 shows the good agreement between the solution of equation (4) by the Adomian method (equations 5 and 6) and the classical Runge-Kutta method.

Figures 2 and 3 show the gravity wave spectra calculated from Eq. (6) (solid line) and calculated from linear solution of Eq. (4) (dotted line), obtained considering $\beta = \text{constant}$. The Figure 2 indicates that the linear solution of Eq. (4) is a good approximation of solution of Eq. (4) for small height (< 4 km). However, the Figure 3 shows that the linear solution is not a good approximation of solution of Eq. (4) for height > 4 km. The Figure 3 (50 km) shows that the linear solution differs completely from the solution of Eq. (4) when considering the nonlinearity of the problem (Eqs. 5 and 6).

The Eq. (4) is nonlinear due to the parameterization of coefficient β expressed in Eq. (2). It is the coefficient of nonlinear damping due to interactions of the component m_R with other waves in the spectrum. This term is essentially nonlinear. Its linearization leads to a completely incorrect solution in height above of the 50 km. Therefore, the linear solution cannot be used to correctly describe the spectrum of kinetic energy when considering height above 4 km (Figures 2 and 3).

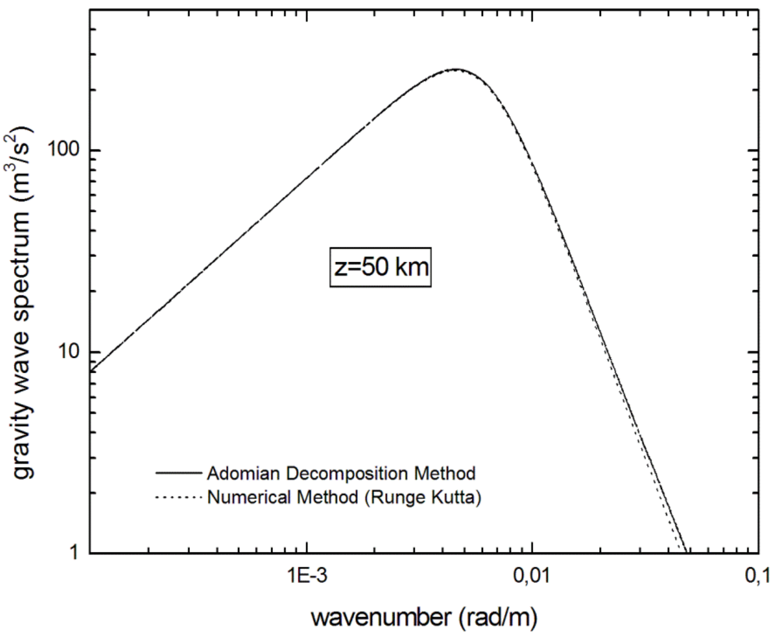

Figure 1. Gravity wave spectra calculated from Eqs. 5 and 6 (solid line) and with the Runge Kutta Numerical Method (dotted line) for the height z = 50 km.

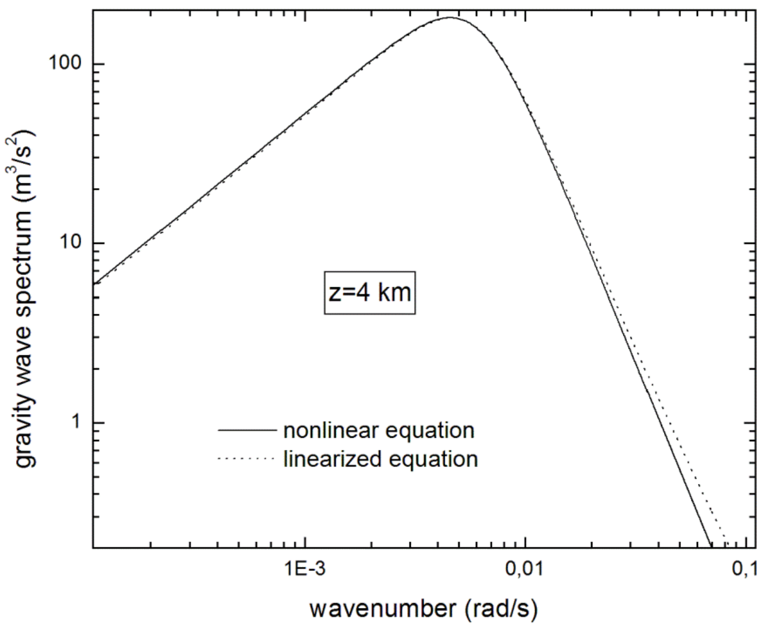

Figure 2. Gravity wave spectra calculated from Eqs. 5 and 5 (solid line) and with linearized solution of Eq. (4) (dotted line) for the height z = 4 km.

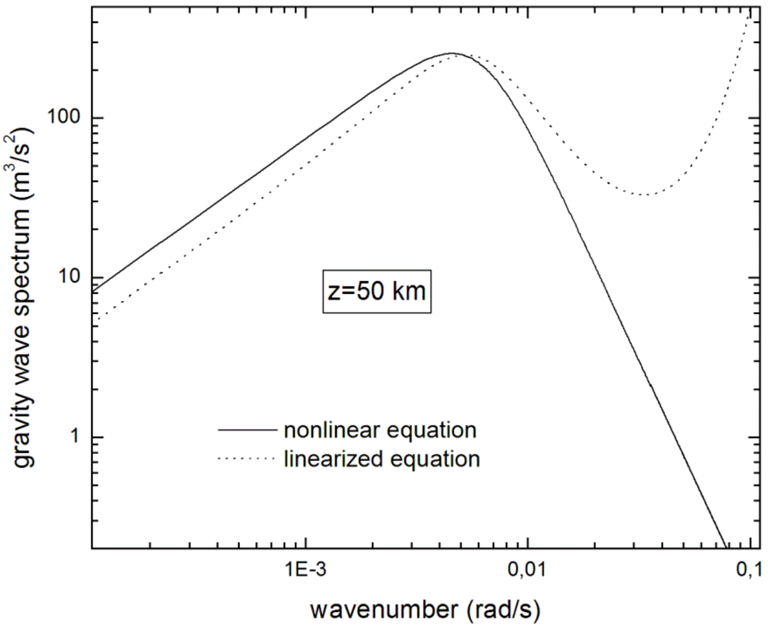

Figure 3. Gravity wave spectra calculated from Eqs 5 and 6 (solid line) and with linearized solution of Eq. (4) (dotted line) for the height z = 50 km.

5. Conclusions

In this study, the nonlinear equation of the evolution of gravity wave spectra with height *z* was analytically solved without linearization by the decomposition method, so the nonlinear nature of problem was preserved. Therefore, mathematically, the errors found are only due to the parameterization used. As a test, we used a spectral parameterization scheme for calculating gravity wave momentum deposition in the middle atmosphere proposed by Medvedev and Klaassen (1995, 2000). This proposed parameterization involves nonlinear wave interactions. Our results indicate that the linear solution of the resultant equation is a good approximation of the solution only for small height (< 4 km). However, the linear solution is not a good approximation of the solution of the resultant equation for height > 4 km, because the linearization of the beta coefficient leads to a solution that does not correctly describe the kinetic energy spectra. The discrepancies depend not on the solution of the nonlinear equation, but on the equation itself, which it is only a reality model. In the model proposed by Medvedev and Klaassen (1995, 2000) the layer height was discretized in sub-intervals of the 0.5 km, in such a manner that inside each interval a linear equation is resolved considering average value of the quantities. This is an approximation used when you cannot solve the resulting equations from the parameterization employed. Taking into account the analytical feature and fast numerical convergence of the solution, besides the fact that this sort of solution is not found in the literature for this problem, we are confident to affirm that the proposed solution is a promising technique to handle realistic physical problems. In view of the potential usefulness of the decomposition method it would be desirable to extend the applicability of the method to test other parameterizations. Work in this direction is in progress.

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